

A CENSUS OF QUADRATIC POST-CRITICALLY FINITE RATIONAL MAPS DEFINED OVER \mathbb{Q}

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ABSTRACT. We find all quadratic post-critically finite (PCF) rational maps defined over \mathbb{Q} . We describe an algorithm to search for possibly PCF maps. Using refinements of the algorithm, we eliminate all but twelve rational maps, all of which are verifiably PCF. We also give a complete description of possible rational preperiodic structures for quadratic PCF maps defined over \mathbb{Q} .

1. INTRODUCTION

Let $\phi(z) \in \mathbb{Q}(z)$ have degree $d \geq 2$. We may regard $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ as a morphism of the projective line. We consider iterates of ϕ :

$$\phi^n(z) = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}(z), \quad \text{and} \quad \phi^0(z) = z.$$

The orbit of a point $\alpha \in \mathbb{P}^1$ is the set $\mathcal{O}_\phi(\alpha) = \{\phi^n(\alpha) \mid n \geq 0\}$.

Rather than studying individual rational maps, we consider equivalence classes of maps under conjugation by $f \in \text{PGL}_2(\mathbb{Q})$; we define $\phi^f = f \circ \phi \circ f^{-1}$. Note that ϕ and ϕ^f have the same dynamical behavior. In particular, f maps the orbit $\mathcal{O}_\phi(\alpha)$ to $\mathcal{O}_{\phi^f}(f(\alpha))$.

Critical points of ϕ are the points $\alpha \in \mathbb{P}^1$ such that $\phi'(\alpha) = 0$ as long as α and $\phi(\alpha)$ are finite. To compute the derivative at the excluded values of α , we use a conjugate map.

Definition 1.1. A rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d \geq 2$ is *postcritically finite* (PCF) if the orbit of each critical point is finite.

A fundamental observation in the study of one-dimensional complex dynamics is that the orbits of the finite set of critical points of ϕ largely determines the dynamics of ϕ on all of \mathbb{P}^1 . So the study of PCF maps has a long history in complex dynamics, including Thurston's topological characterization of these maps in the early 1980s and continuing to the present day. In [2], for example, the authors find exactly one representative from each conjugacy class of nonpolynomial hyperbolic PCF rational maps of degree 2 and 3 in which the post-critical set — the forward orbit of the critical points, excluding the points themselves — contains no more than four points.

In [12], Silverman advances the idea of PCF maps as a dynamical analog of abelian varieties with complex multiplication, suggesting that these maps may be of special interest in arithmetic dynamics as well. Inspired by this idea and by [2], we

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have our main result below. Compare this to the statement that, up to isomorphism over \mathbb{Q} , there are exactly thirteen elliptic curves E/\mathbb{Q} with complex multiplication.

Theorem 1.2. *There are exactly twelve conjugacy classes of quadratic PCF maps defined over \mathbb{Q} :*

$$\begin{array}{ll}
 (1) & z^2 \\
 (2) & 1/z^2 \\
 (3) & z^2 - 2 \\
 (4) & z^2 - 1 \\
 (5) & \frac{1}{2(z-1)^2} \\
 (6) & \frac{1}{(z-1)^2} \\
 (7) & \frac{-1}{4z^2 - 4z} \\
 (8) & \frac{-4}{9z^2 - 12z} \\
 (9) & \frac{2}{(z-1)^2} \\
 (10) & \frac{2z+1}{4z-2z^2} \\
 (11) & \frac{-2z}{2z^2-4z+1} \\
 (12) & \frac{3z^2-4z+1}{1-4z}
 \end{array}$$

Of these, the first four were well-known to researchers in both complex and arithmetic dynamics. Maps (5)–(8) appeared in [2]. Maps (9)–(12) did not appear in [2] because they fail to fit either the criterion of hyperbolicity or the post critical set is too large. One major contribution of this work is the fact that this list is exhaustive.

A fundamental question in arithmetic dynamics is classifying rational maps by the structure of their rational preperiodic points. In [10], Poonen undertakes this task for quadratic polynomials defined over \mathbb{Q} , subject to the condition that no rational point is on a cycle of length greater than 3. The first author [5] gives a classification for rational maps with nontrivial PGL_2 stabilizer, subject to a similar condition.

Given the comprehensive list in Theorem 1.2, we are able to describe all possible rational preperiodic structures for quadratic PCF maps defined over \mathbb{Q} with no additional hypotheses. Difficulty arises only for the first two maps, which have nontrivial twists. We are able to conclude the following.

Theorem 1.3. *A quadratic PCF map defined over \mathbb{Q} has at most six rational preperiodic points.*

Given the parallels between the set of rational preperiodic points for a rational map and the torsion subgroup of an abelian variety $A(\mathbb{Q})$ — see [12, page 111], for example — this result and the preperiodic structures given in Section 6 are analogs of the comprehensive list of torsion subgroups for CM elliptic curves E/\mathbb{Q} in [9].

1.1. Outline. We begin in Section 2 with a few results that form the basis of our algorithm for finding quadratic rational PCF maps. In Section 3, we give the the algorithm, and Section 4 describes the resulting list of quadratic PCF maps defined over \mathbb{Q} . Section 5 details our method finding rational preperiodic structures for the two PCF maps with nontrivial twists. We conclude with a comprehensive description of preperiodic structures for all quadratic PCF maps defined over \mathbb{Q} in Section 6.

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The algorithm in Section 3 and complete list of PCF maps constitutes a portion of the second author's Ph.D. thesis.

2. BACKGROUND

The following theorem, derived from a result of Manes and Yasufuku [7], is used to iterate through equivalence classes of quadratic rational maps. It applies to those maps with trivial PGL_2 stabilizer. Maps with nontrivial stabilizer are addressed in Section 5.

Theorem 2.1. *Let K be a field with characteristic different from 2 and 3. Let $\psi(z) \in K(z)$ have degree 2, and let $\lambda_1, \lambda_2, \lambda_3 \in \overline{K}$ be the multipliers of the fixed points of ψ (counted with multiplicity). Then $\psi(z)$ is conjugate over K to the map*

$$\phi(z) = \frac{2z^2 + (2 - \sigma_1)z + (2 - \sigma_1)}{-z^2 + (2 + \sigma_1)z + 2 - \sigma_1 - \sigma_2} \in K(z),$$

where σ_1 and σ_2 are the first two symmetric functions of the multipliers. Furthermore, no two distinct maps of this form are conjugate to each other over \overline{K} .

The following result from [1] makes it feasible to enumerate all quadratic PCF maps. Here, $H(\lambda)$ is the standard multiplicative height. (See [11, Section 3.1] for background on heights.)

Lemma 2.2 (Corollary 1.3 in [1]). *Let $\phi(z) \in \overline{\mathbb{Q}}(z)$ have degree 2, suppose that ϕ is PCF, and let λ be the multiplier of any fixed point of ϕ . Then $H(\lambda) \leq 4$.*

Using this, we can derive explicit height bounds for σ_1 and σ_2 .

Proposition 2.3. *Let $\phi(z) \in \overline{\mathbb{Q}}$ be a degree 2 PCF map, and suppose that σ_1 and σ_2 are the first and second symmetric functions on the multipliers of the fixed points. Then $H(\sigma_1) \leq 192$ and $H(\sigma_2) \leq 12288$.*

Proof. We simplify notation by setting $d = [K : \mathbb{Q}]$ for K any field of definition of the fixed point multipliers. By the triangle inequality:

$$|\sigma_1|_v = |\lambda_1 + \lambda_2 + \lambda_3|_v \leq \begin{cases} \max\{|\lambda_1|_v, |\lambda_2|_v, |\lambda_3|_v\} & \text{for each finite place} \\ 3 \max\{|\lambda_1|_v, |\lambda_2|_v, |\lambda_3|_v\} & \text{for each infinite place.} \end{cases}$$

For an extension of degree d , there are at most d infinite places, so

$$\begin{aligned} H(\sigma_1) &= \prod_{v \in M_K} (\max\{|\sigma_1|_v, 1\}^{n_v})^{1/d} \leq 3 \prod_{v \in M_K} \left(\max_{1 \leq i \leq 3} \{|\lambda_i|_v, 1\}^{n_v} \right)^{1/d} \\ &\leq 3 \prod_{v \in M_K} (\max\{|\lambda_1|_v, 1\}^{n_v} \cdot \max\{|\lambda_2|_v, 1\}^{n_v} \cdot \max\{|\lambda_3|_v, 1\}^{n_v})^{1/d} \\ &= 3H(\lambda_1)H(\lambda_2)H(\lambda_3) \leq 3 \cdot 4^3 = 192. \end{aligned}$$

The proof for the bound on σ_2 follows similarly:

$$\begin{aligned}
H(\sigma_2) &= H(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \leq 3 \prod_{v \in M_K} \left(\max_{\substack{i \neq j \\ 1 \leq i, j \leq 3}} \{|\lambda_i\lambda_j|_v, 1\}^{n_v} \right)^{1/d} \\
&\leq 3 \prod_{v \in M_K} (\max\{|\lambda_1\lambda_2|_v, 1\}^{n_v} \cdot \max\{|\lambda_2\lambda_3|_v, 1\}^{n_v} \cdot \max\{|\lambda_1\lambda_3|_v, 1\}^{n_v})^{1/d} \\
&= 3H(\lambda_1\lambda_2)H(\lambda_2\lambda_3)H(\lambda_1\lambda_3) \leq 3 \cdot 4^6 = 12288. \quad \square
\end{aligned}$$

This height bound coupled with the normal form of Theorem 2.1 allows us to:

- iterate through possible rational values of σ_1 and σ_2 ,
- form a unique rational map from each equivalence class, and
- test if that map is PCF.

The final step of the algorithm sketched above relies on the following two theorems. For notation: K is a local field with nonarchimedean absolute value $|\cdot|_v$, R is the ring of integers of K , \mathfrak{p} is the maximal ideal of R , $k = R/\mathfrak{p}$ is the residue field, and \sim is the reduction map modulo \mathfrak{p} .

Theorem 2.4 (Theorem 2.21 in [11]). *Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational function of degree $d \geq 2$ defined over K . Assume that ϕ has good reduction, let $P \in \mathbb{P}^1(K)$ be a periodic point of ϕ , and define the following quantities:*

n The exact period of P for the map ϕ .

m The exact period of \tilde{P} for the map $\tilde{\phi}$.

r The order of $\lambda_{\tilde{\phi}}(\tilde{P}) = (\tilde{\phi}^m)'(\tilde{P})$ in k^* . (Set $r = \infty$ if $\lambda_{\tilde{\phi}}(\tilde{P})$ is not a root of unity.)

p The characteristic of the residue field k .

Then n has one of the following forms:

$$n = m \quad \text{or} \quad n = mr \quad \text{or} \quad n = mrp^e.$$

Theorem 2.5 (Theorem 2.28 in [11]). *We continue with the notation and assumptions from Theorem 2.4. We further assume that K has characteristic 0 and we let $v : K^* \rightarrow \mathbb{Z}$ be the normalized valuation on K . If the period n of $P \in \mathbb{P}^1(K)$ has the form $n = mrp^e$, then the exponent e satisfies*

$$p^{e-1} \leq \frac{2v(p)}{p-1}.$$

Let $\phi(z) \in \mathbb{Q}(z)$ with critical points γ_1, γ_2 . To apply Theorem 2.4, we consider ϕ to be defined over \mathbb{Q}_p for p a prime of good reduction with $\gamma_i \in \mathbb{Q}_p$. If ϕ is PCF, then some iterate $\phi^j(\gamma_i)$ has exact period n . Since the \mathbb{F}_p -orbit $\mathcal{O}_{\tilde{\phi}}(\tilde{\gamma}_i)$ is necessarily finite, some iterate $\tilde{\phi}^k(\tilde{\gamma}_i)$ has exact period m . Theorem 2.4 gives us a set of possible values for n based on the more-easily computed value m .

In the algorithm of Section 3, we find these sets for several primes p and intersect them. An empty intersection shows that a global period n cannot exist (i.e. that

the map is not PCF). Since we are working in \mathbb{Q}_p , $\nu(p) = 1$, so Theorem 2.5 allows us to assert $e = 0$ when $p \neq 2$, and $e \in \{0, 1\}$ when $p = 2$. In the algorithm, we exclude $p = 2$ from consideration, so we have the following possibilities for n :

$$n = m \quad \text{or} \quad n = mr.$$

3. ALGORITHM

The algorithm used was written in Sage [13] and uses a subroutine for finding orbits from the ProjSpace package developed at ICERM [3]. Code for this algorithm will be made available in the arXiv distribution of the article.

We first sketch the flow of the algorithm. Iterating over all degree 2 rational maps ϕ with trivial PGL_2 stabilizer:

- (1) Find the resultant R of ϕ .
- (2) Make a list of the first n primes.
- (3) Remove from the list any prime p such that $\gcd(R, p) > 1$; these are primes of bad reduction.
- (4) Find the critical points of ϕ .
- (5) For each critical point γ , run over the list of primes to find the length of the cycle into which $\tilde{\gamma}$ falls, m_p . Find the multiplier of this cycle r_p . Create a list of possible global periods $L_p = \{m_p, m_p r_p\}$.
- (6) Iteratively intersect the lists L_p found in (5) and discard the map if the intersection is empty at any point.

Algorithm 1 — check_periods

Input:

- a degree 2 rational map $\phi(z) \in \mathbb{Z}(z)$
- a critical point γ of ϕ
- a list P of good primes

Output: TRUE if there is a possible global period for the orbit containing γ that agrees with the period in \mathbb{F}_p for every prime in $p \in P$

let p_1 be the first prime p_1 in P

find the tail and period m_{p_1} for the orbit of $\tilde{\gamma}$ in \mathbb{F}_{p_1}

create the list $L = \{m_{p_1}, m_{p_1} r_{p_1}\}$ of possible global periods for γ

for remaining p in P :

find the tail and period m_p for the orbit of $\tilde{\gamma}$ in \mathbb{F}_p

create list $L_p = \{m_p, m_p r_p\}$ of possible global periods for γ

let L be the intersection of L and L_p

if L is empty:

return FALSE

return TRUE

In Algorithm 2, if the critical points are irrational we only check the possible global periods for one of them. Since $\phi(z) \in \mathbb{Q}(z)$ is quadratic, the critical points γ_1 and γ_2 are Galois conjugates, so the same is true of $\phi^i(\gamma_1)$ and $\phi^i(\gamma_2)$ for every $i \geq 0$. Therefore, the orbits of γ_1 and γ_2 are either both finite or both infinite,

and if they are finite they will terminate in cycles of the same length. We take advantage of this symmetry to speed up the algorithm.

Algorithm 2 — **is_PCF** filters out maps ϕ which are not PCF

Input:

- a degree 2 rational map $\phi(z) \in \mathbb{Q}(z)$
- the number of primes to test

Output: TRUE, if ϕ passes the filter

define P a list of odd primes of good reduction

define γ_1, γ_2 the critical points of ϕ :

if $\gamma_1 \in \mathbb{Q}$:

 for $i = 1, 2$:

 if not **check_periods**(ϕ, γ_i, P):
 return FALSE

else (if $\gamma_1 \notin \mathbb{Q}$):

 modify P to exclude primes where $\gamma_1 \notin \mathbb{Q}_p$

 if not **check_periods**(ϕ, γ_1, P):
 return FALSE

return TRUE

In Algorithm 3, $\phi(\sigma_1, \sigma_2)$ refers to the normal form given in Theorem 2.1. The condition on the stabilizer is a simple polynomial calculation in the variables (σ_1, σ_2) . See [7] for details.

Algorithm 3 — Executes Algorithm 2 (**is_PCF**) up to height bound

Input: [none]

Output: list of possible PCF maps

for $\sigma_2 \in \mathbb{Q}$ of height ≤ 12288 :

 for $\sigma_1 \in \mathbb{Q}$ of height ≤ 192 :

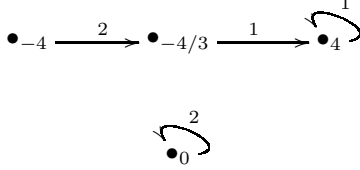
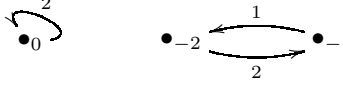
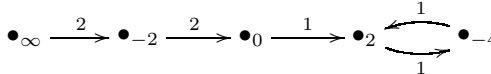
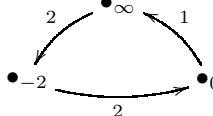
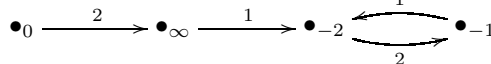
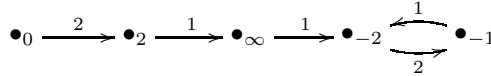
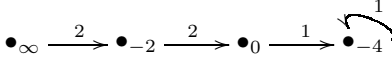
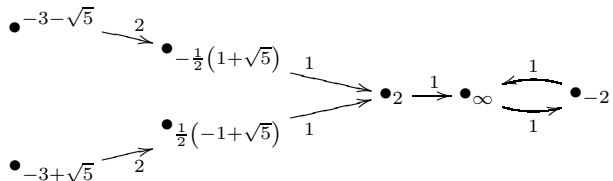
 if $\phi(\sigma_1, \sigma_2)$ has trivial stabilizer:
 if **is_PCF**($\phi, 25$):
 print ϕ

Note that this algorithm filters out maps which are certainly not PCF, but does not guarantee that the maps which remain are PCF. The initial run of the algorithm used the first $n = 25$ primes, and it eliminated all but a handful of maps. Of these, false positives were removed by increasing the number of primes used to seed the algorithm to $n = 100$. The remaining ten maps were verified to be PCF by simply iterating the map at each critical point; these maps are described in Section 4.

4. PCF MAPS WITHOUT SYMMETRIES

Table 1 lists the output of our algorithm: all quadratic PCF maps defined over \mathbb{Q} with trivial PGL_2 stabilizer. We also give the critical point portraits for these maps. In these graphs, an arrow from P to Q indicates that $\phi(P) = Q$; an integer over

the arrow indicates the ramification index of the map at that point. In particular, the critical points are the initial points for arrows where the integer is 2. The final column offers a conjugate map in a simpler form.

$\phi(z)$	Critical portrait	Conjugate map
$\frac{2z^2}{-z^2 + 4z + 8}$		$z^2 - 2$
$\frac{2z^2}{-z^2 + 4z + 4}$		$z^2 - 1$
$\frac{2z^2 + 8z + 8}{-z^2 - 4z + 4}$		$\frac{1}{2(z-1)^2}$
$\frac{2z^2 + 8z + 8}{-z^2 - 4z}$		$\frac{1}{(z-1)^2}$
$\frac{2z^2 + 4z + 4}{-z^2}$		$\frac{-1}{4z^2 - 4z}$
$\frac{6z^2 + 8z + 8}{-3z^2 + 4z + 4}$		$\frac{-4}{9z^2 - 12z}$
$\frac{2z^2 + 8z + 8}{-z^2 - 4z - 2}$		$\frac{2}{(z-1)^2}$
$\frac{2z^2 + 4z + 4}{-z^2 + 4}$		$\frac{2z+1}{4z-2z^2}$

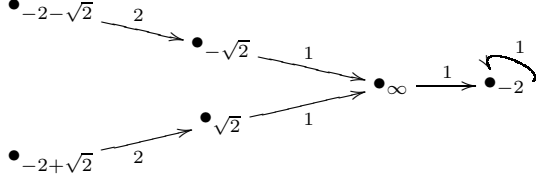
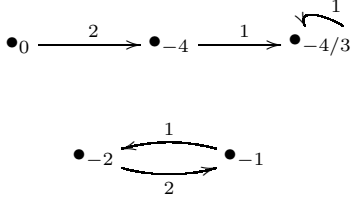
$\frac{2z^2 + 4z + 4}{-z^2 + 2}$		$\frac{-2z}{2z^2 - 4z + 1}$
$\frac{6z^2 + 16z + 16}{-3z^2 - 4z - 4}$		$\frac{3z^2 - 4z + 1}{1 - 4z}$

Table 1: All quadratic PCF maps defined over \mathbb{Q} with trivial PGL_2 stabilizer

Remark 4.1. This list of maps raises some questions.

- All maps except the sixth one and the last one — the maps $\frac{-4}{9z^2 - 12z}$ and $\frac{3z^2 - 4z + 1}{1 - 4z}$ — satisfy $\sigma_1 \in \{\pm 2, -6\}$. (This is also true of the maps with nontrivial stabilizer described in Section 5.) The line $\sigma_1 = 2$ in the moduli space of quadratic rational maps corresponds to the quadratic polynomials. What (if anything) is special about these other two lines?
- Similarly, all maps except the sixth one and the last one correspond to integer values of (σ_1, σ_2) . What is special about these the two anomalous maps?
- For the two anomalous maps we have $(\sigma_1, \sigma_2) = (-\frac{2}{3}, \frac{4}{3})$ and $(\sigma_1, \sigma_2) = (-\frac{10}{3}, \frac{20}{3})$. In other words, the symmetric functions of the multipliers have denominator at most 3 for all quadratic PCF maps defined over \mathbb{Q} . Is there some general phenomenon here that extends to maps defined over number fields?

From [11, Proposition 4.73], morphisms with trivial PGL_2 stabilizer have no nontrivial twists. That is, any quadratic PCF map defined over \mathbb{Q} with trivial stabilizer must be conjugate to one of the ten maps above, and the conjugacy must also be defined over \mathbb{Q} . Hence the rational preperiodic structures for these maps are invariant within the conjugacy class. We use [3] to find them all. (The algorithm to find rational preperiodic structures, like the one used to find potentially PCF maps, is based on Theorem 2.4.) The results appear in Table 2.

5. PCF MAPS WITH SYMMETRIES

Quadratic rational maps with nontrivial PGL_2 stabilizer — called variously *maps with symmetries*, the *symmetry locus* in the moduli space, or maps with *nontrivial*

automorphisms — have been extensively studied. In [8], Milnor described the symmetry locus for quadratic rational maps; the first author investigated the arithmetic of these maps in [5, 6]. Jones and Manes found a height bound on PCF maps with symmetries and used that bound to show that over \mathbb{Q} , the only maps meeting these criteria must be conjugate to either $\psi_1(z) = z^2$ or $\psi_2(z) = 1/z^2$ [4, Proposition 5.1].

Unlike the six maps described in Section 4, these two maps have nontrivial twists. That is, there are infinitely many \mathbb{Q} conjugacy classes within each of these two $\bar{\mathbb{Q}}$ conjugacy classes of maps. The different \mathbb{Q} conjugacy classes may have very different structures for their rational preperiodic points. In this section, we find all of the possible rational preperiodic structures for these two conjugacy classes.

To analyze preperiodic points, we need the following definition.

Definition 5.1. If a point $\alpha \in \mathbb{P}^1(\mathbb{Q})$ enters a cycle of least period m after n iterations (i.e. if $\phi^n(\alpha)$ has period m with n and m minimal), then α is called a rational point of type m_n .

Throughout this section, ζ_n represents a primitive n^{th} root of unity.

5.1. Maps conjugate to $\psi_1(z) = z^2$. Twists of ψ_1 are described completely in [5]. They are given by

$$\phi_b(z) = \frac{z}{2} + \frac{b}{z},$$

where $b \neq 0$ is defined up to squares in \mathbb{Q} .

Applying propositions from [5], we easily conclude:

- (1) ϕ_b always has a rational fixed point at infinity and a rational point of type 1_1 at 0 (Propositions 1 and 5).
- (2) ϕ_b has finite rational fixed points if and only if $b = 1 - k$ up to squares (Proposition 1). So take $b = 1/2$ to get the map

$$\phi_{1/2}(z) = \frac{z^2 + 1}{2z}.$$

In this case, there are no additional points of type 1_1 (Proposition 5).

- (3) ϕ_b has rational points of primitive period 2 if and only if $b = -(k + 1)$ up to squares (Proposition 2). So we take $b = -3/2$ to get the map

$$\phi_{-3/2}(z) = \frac{z^2 - 3}{2z}.$$

In this case, we also have two rational points of type 2_1 (Proposition 5) but no rational points of type 2_n for $n > 1$ (Proposition 8) and no finite rational fixed points (Proposition 9).

- (4) ϕ_b cannot have rational points of primitive period 3 or 4 (Theorems 3 and 4). This will also follow Theorem 5.2 below.
- (5) ϕ_b has rational points of type 1_2 if and only if $b = -k$ up to squares, so we take $b = -1/2$ to get the map

$$\phi_{-1/2}(z) = \frac{z^2 - 1}{2z}.$$

In this case, there are no finite rational fixed points (Proposition 6) and no rational points of period 2 (Proposition 9).

- (6) ϕ_b cannot have rational points of type 1_n for $n \geq 3$ (Propositions 7 and 8).

The description above yields four possible rational preperiodic structures, shown in Table 3. In order to claim we have a complete description of the possible rational preperiodic structures, we need the following result.

Theorem 5.2. *Let*

$$\phi_b(z) = \frac{z}{2} + \frac{b}{z}.$$

Then ϕ has no rational point of primitive period $n > 2$.

Proof. Consider a point $\alpha \in \mathbb{Q}$ so that α is periodic for $\phi_b(z)$. Let

$$f(z) = z\sqrt{\frac{b}{2}}, \quad \text{so} \quad \phi_b^f(z) = \phi_{1/2}(z) = \frac{z^2 + 1}{2z}.$$

Then we have $f^{-1}(\alpha) = \alpha\sqrt{\frac{2}{b}}$ is periodic for $\phi_{1/2}(z)$.

Now let $g = (z + 1)/(-z + 1)$. It's a simple matter to check that $\phi_{1/2}^g(z) = z^2$, so that $g^{-1}(f^{-1}(\alpha)) \in \mathbb{Q}[\sqrt{b/2}]$ is periodic for z^2 .

We will now categorize periodic points for $\psi_1(z) = z^2$ that lie in quadratic fields, showing that none of them have period of length more than 2. The result will follow.

The map ψ_1 has a totally ramified fixed point at ∞ . Any finite periodic point of $\psi_1(z) = z^2$ is a root of $z^{2^n} - z$, so it is either 0 or a root of $z^{2^n-1} - 1$, i.e. a root of unity. Since we seek periodic points that lie in quadratic fields, we can restrict our search to roots of unity that lie in quadratic fields, namely $\{\pm 1, \pm i, \zeta_3, \zeta_3^{-1}, \zeta_6, \zeta_6^{-1}\}$.

A computation verifies that the preperiodic structures for ψ_1 containing these points are the ones shown in Figure 1. So the only quadratic periodic points have period 1 or 2 as desired. \square



FIGURE 1. All possible quadratic periodic points for $\psi(z) = z^2$.

5.2. Maps conjugate to $\psi_2(z) = 1/z^2$. From [7], all such maps are conjugate over \mathbb{Q} to a map of the form

$$\theta_{d,k}(z) = \frac{kz^2 - 2dz + dk}{z^2 - 2kz + d}, \quad \text{with } k \in \mathbb{Q}, d \in \mathbb{Q}^*, \text{ and } k^2 \neq d. \quad (5.1)$$

Conjugating this map by

$$f(z) = \frac{\sqrt{d}z + \sqrt{d}}{1 - z}$$

yields

$$\theta_{d,k}^f(z) = \frac{t}{z^2} \quad \text{where } t = \frac{k - \sqrt{d}}{k + \sqrt{d}}.$$

Conjugating this by $g(z) = t^{1/3}z$ gives

$$\left(\theta_{d,k}^f\right)^g(z) = \frac{1}{z^2}.$$

If $\alpha \in \mathbb{Q}$ is preperiodic for $\theta_{d,k}$, then $\beta = g^{-1}f^{-1}(\alpha) \in \mathbb{Q}(t^{1/3})$ is a preperiodic point for $\psi_2(z)$. Since $[\mathbb{Q}(\beta) : \mathbb{Q}] \leq 6$, we may find all rational preperiodic structures for this family of maps by describing preperiodic points for ψ_2 of degree at most six. Conjugating these points to lie in the rationals, we will find a map in the family with specified rational preperiodic points or show that none exists.

Lemma 5.3. *All preperiodic points for $\psi_2(z) = 1/z^2$ of degree at most six are given in Figures 2–5.*

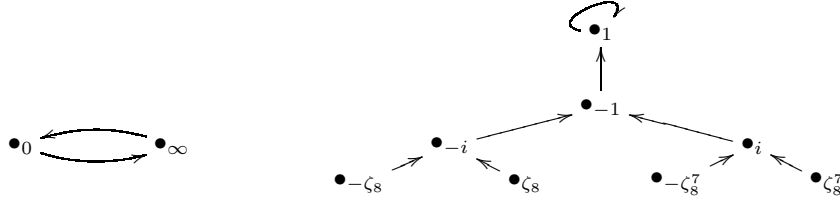


FIGURE 2. $\psi_2(z) = 1/z^2$: Two cycle and one fixed point.



FIGURE 3. $\psi_2(z) = 1/z^2$: Two additional fixed points.

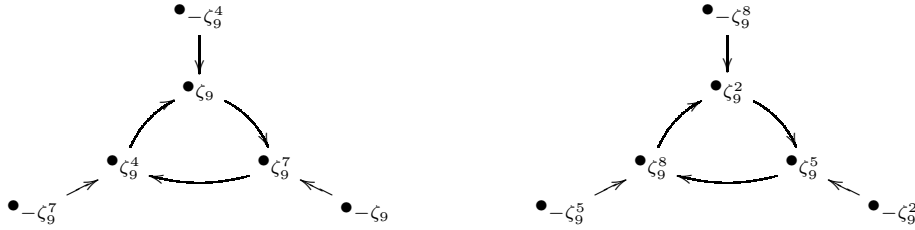
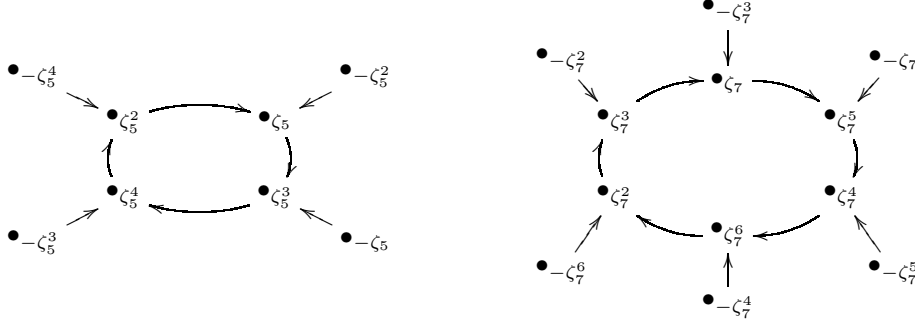


FIGURE 4. $\psi_2(z) = 1/z^2$: Two three-cycles.

Proof. For n even, $\psi_2^n(z) = z^{2^n}$, so the points with period dividing n are 0 , ∞ , and $(2^n - 1)^{\text{st}}$ roots of unity. For n odd, $\psi_2^n(z) = z^{-2^n}$, so the points with period dividing n are $(2^n + 1)^{\text{st}}$ roots of unity. Hence all strictly periodic points other than 0 and ∞ are roots of unity of odd order.

FIGURE 5. $\psi_2(z) = 1/z^2$: A four-cycle and a six-cycle.

The only roots of unity of odd order with degree no more than 6 are powers of $\{1, \zeta_3, \zeta_5, \zeta_7, \zeta_9\}$. We may find their periodic structures by iterating ψ_2 with the appropriate seed values.

Let β be a preperiodic point for ψ_2 . Then $[\mathbb{Q}(\beta) : \mathbb{Q}] \leq 6$ if and only if all powers of β also satisfy $[\mathbb{Q}(\beta^n) : \mathbb{Q}] \leq 6$. In particular, the orbit of β lands in some cycle, and the points of that cycle have degree no more than six. Hence we can find all preperiodic points for ψ_2 having degree no more than six by finding preimages of the periodic points described above, and continuing until the field generated by the preimages has degree greater than six.

It is a simple matter to verify that this process yields the diagrams given. \square

Proposition 5.4. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2 . Then ϕ has no points of type 2_n for $n \geq 1$. For $m \neq 2$, ϕ has the same number of rational points of type m_1 as it has rational points of primitive period m .*

Proof. The critical points of ψ_2 lie on a two cycle, and this property is preserved under conjugation. Therefore each critical point is also a critical value, so if the critical points of ϕ are $\{\gamma_1, \gamma_2\}$ we have $\phi^{-1}(\gamma_i) = \{\gamma_j\}$ for $i \neq j$. Hence ϕ has no points of type 2_1 and it follows that ϕ has no points of type 2_n for $n \geq 1$.

Let α be a rational point of primitive period m for ϕ . Then all points on the m -cycle containing α are also rational since $\phi(z) \in \mathbb{Q}(z)$. Therefore the quadratic $\phi(z) = \alpha$ has one rational root. Since $m \neq 2$, α is not one of the critical values of ϕ by the argument above. Hence, the quadratic $\phi(z) = \alpha$ has two distinct roots and both must be rational. That is, there is a rational point β not on the m -cycle satisfying $\phi(\beta) = \alpha$, and β is a point of type m_1 . \square

Proposition 5.5. *Let $\phi(z) \in \mathbb{Q}(z)$ be conjugate to ψ_2 . If ϕ has a rational two-cycle then it may have either no rational fixed points or one rational fixed point. In either case, it has no other rational preperiodic points except the required point of type 1_1 .*

Proof. From [7, Lemma 5.1], we see that ϕ has a rational two-cycle if and only if it is conjugate over \mathbb{Q} to $\theta_t(z) = \frac{t}{z^2}$ for some $t \in \mathbb{Q}^\times$. Solving $\theta_t(z) = z$, we see that there is a rational fixed point if and only if $t \in (\mathbb{Q}^\times)^3$, and all such maps are conjugate over \mathbb{Q} .

Furthermore, if $f(z) = t^{1/3}z$, then $\theta_t^f(z) = \psi_2$. Applying f^{-1} to the preperiodic structures given in Lemma 5.3, we find no other rational preperiodic points. \square

By Proposition 5.4, we have only two rational preperiodic structures for maps conjugate to $\phi_2(z)$ that contain rational points of primitive period 2. These are the first two maps represented in Table 4.

Proposition 5.6. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2 . Suppose ϕ has no rational points of period $n > 1$. Then ϕ has one of the following rational preperiodic structures:*

- (a) ϕ has no rational fixed points (hence no rational preperiodic points at all);
- (b) ϕ has exactly one rational fixed point and one point of type 1_1 but no other rational preperiodic points;
- (c) ϕ has exactly one rational fixed point, one rational point of type 1_1 , and two rational points of type 1_2 , with no other rational preperiodic points; or
- (d) ϕ has exactly three rational fixed points and three rational points of type 1_1 , with no other rational preperiodic points.

Proof. Choosing $k = 1$ and $d = 2$ in the normal form from equation (5.1) yields the map

$$\frac{z^2 - 4z + 2}{z^2 - 2z + 2}.$$

One can check computationally that this map has no rational points of primitive period 1, 2, 3, 4, or 6. By Lemma 5.3, these are the only possibilities.

Choosing $k = 0$ and $d = 2$ in the normal form from equation (5.1) yields the map

$$-\frac{4z}{z^2 + 2}.$$

One can check computationally that this map has fixed point 0 and no other rational points of primitive period 1, 2, 3, 4, or 6. By Lemma 5.3, these are the only possibilities. We also have $\infty \mapsto 0$, a rational point of type 1_1 . The preimages of ∞ are not rational, so there are no other rational preperiodic points.

Beginning with the preperiodic structure described in Lemma 5.3, we see that conjugating ϕ_2 by any $f \in \text{PGL}_2$ which maps three arbitrary rational points to 1, i , and $-i$ creates a map with rational type 1_2 points. Choose

$$f(z) = \frac{-1 + iz}{-z + i} \quad \text{which yields } \psi_2^f(z) = \frac{-z^2 + 2z + 1}{z^2 + 2z - 1}.$$

One can check computationally that this map has no rational point of period 2, 3, 4, or 6. The only rational fixed point is 1; -1 is a type 1_1 point; and 0 and ∞ are type 1_2 points. There are no rational type 1_3 points.

Again, beginning with the preperiodic structure described in Lemma 5.3, we see that conjugating ϕ_2 by any $f \in \text{PGL}_2$ which maps three arbitrary rational points to 1, ζ_3 , and ζ_3^2 yields a map with three rational fixed points. Choose

$$f(z) = \frac{\zeta_3^2 z + 1}{z - (1 + \zeta_3)} \quad \text{which yields } \psi_2^f(z) = -\frac{(z - 2)z}{2z - 1}.$$

This map has fixed points at 0, 1, and ∞ and the corresponding rational type 1_1 points. One can check computationally that this point has no rational point of period 2, 3, 4, or 6, and no rational type 1_2 points.

We have shown that each of the possibilities listed are possible for maps conjugate to ψ_2 . It remains to check that no other possibilities exist.

Since ϕ is defined over \mathbb{Q} , the cubic polynomial $\phi(z) = z$ has either zero, one, or three rational roots. Hence we cannot have exactly two rational fixed points.

If a map ϕ is conjugate to ψ_2 and has rational points of type 1_3 , then it is conjugate over \mathbb{Q} to a map with \mathbb{Q} to a map with a fixed point at 1 and the type 1_2 points at 0 and ∞ . We found such a map above, and it does not have rational type 1_3 points.

Similarly, if a map ϕ is conjugate to ψ_2 and has three rational fixed points and rational points of type 1_2 , then it is conjugate over \mathbb{Q} to a map with one fixed point at 1 and its type 1_2 points at 0 and ∞ . We found such a map above, and it does not have additional rational fixed points.

We have now exhausted all possibilities. \square

By Proposition 5.6, the third through sixth rational preperiodic structures in Table 4 are the only ones possible for maps conjugate to ψ_2 that have no rational points of primitive period $n > 1$.

Proposition 5.7. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2 . Suppose ϕ has a rational point of period 3. Then ϕ has exactly three such points and three points of type 3_1 . The map ϕ has no other rational preperiodic points.*

Proof. If ϕ is conjugate to ψ_2 and has a rational point of period 3, then it is conjugate over \mathbb{Q} to a map with the three cycle $0 \mapsto 1 \mapsto \infty \mapsto 0 \mapsto \dots$. This conjugacy completely specifies the map. Given the preperiodic structure described in Lemma 5.3, we may begin with $f \in \text{PGL}_2$ which maps 0, 1, and ∞ to ζ_9 , ζ_9^7 , and ζ_9^6 . This is

$$f = \frac{\zeta_9^4 z + \zeta_9^7}{z + \zeta_9^6} \quad \text{which yields } \psi_2^f(z) = \frac{2z - 1}{z^2 - 1}.$$

One may verify computationally that this map has the desired three cycle and no other rational points of period 1, 2, 3, 4, or 6. It has rational type 3_1 points mapping into the three cycle, but the type 3_2 points are not rational. \square

By Proposition 5.7, there is only one rational preperiodic structure for maps conjugate to ψ_2 that have a rational point of primitive period 3. This is the last map in Table 4.

Proposition 5.8. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2 . Then ϕ has no rational points of period $n > 3$.*

Proof. If ϕ has rational points of period 4, then it is conjugate over \mathbb{Q} to a map where three of those points are at 0, 1, and ∞ . Applying Lemma 5.3, we choose $f \in \text{PGL}_2$ mapping these three rational points to three powers of ζ_5 . Conjugating ψ_2 by this map does not yield a map defined over \mathbb{Q} .

The argument for points of period 6 is the same, but using powers of ζ_7 . \square

This completes the classification of rational preperiodic structures for maps defined over \mathbb{Q} and conjugate to ψ_2 .

6. PREPERIODIC STRUCTURES

In this section, we give the complete rational preperiodic structures for quadratic PCF maps defined over \mathbb{Q} .

$\phi(z)$	Rational Preperiodic Points Graph
$z^2 - 2$	
$z^2 - 1$	
$\frac{1}{2(z-1)^2}$	
$\frac{1}{(z-1)^2}$	
$\frac{-1}{4z^2 - 4z}$	
$\frac{-4}{9z^2 - 12z}$	
$\frac{2}{(z-1)^2}$	
$\frac{2z+1}{4z-2z^2}$	
$\frac{-2z}{2z^2 - 4z + 1}$	

$\frac{3z^2 - 4z + 1}{1 - 4z}$	
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Table 2: Rational maps with trivial stabilizer

$\phi_b(z) = \frac{z}{2} + \frac{b}{z}$	Rational Preperiodic Points Graph
$\phi_1(z) = \frac{z}{2} + \frac{1}{z}$	
$\phi_{1/2}(z) = \frac{z}{2} + \frac{1}{2z}$	
$\phi_{-3/2}(z) = \frac{z}{2} - \frac{3}{2z}$	
$\phi_{-1/2}(z) = \frac{z}{2} - \frac{1}{2z}$	

Table 3: Twists of $\psi_1(z) = z^2$

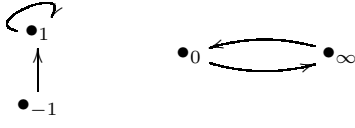
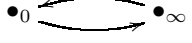
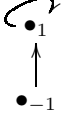
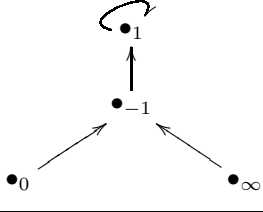
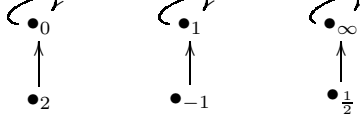
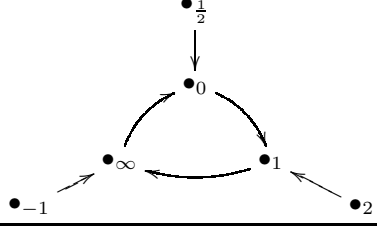
$\phi(z)$	Rational Preperiodic Points Graph
$\frac{1}{z^2}$	
$\frac{2}{z^2}$	
$\frac{z^2 - 4z + 2}{z^2 - 2z + 2}$	
$-\frac{4z}{z^2 + 2}$	
$\frac{-z^2 + 2z + 1}{z^2 + 2z - 1}$	
$-\frac{(z-2)z}{2z-1}$	
$\frac{2z-1}{z^2-1}$	

Table 4: Twists of $\psi_2(z) = 1/z^2$

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